



# How Can One Create a Material with a Prescribed Refraction Coefficient?

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## Abstract

A recipe is given for creating materials with prescribed refraction coefficient. This recipe would be practically applicable if there is method for preparing small particles with prescribed boundary impedance. The problem of practical preparing small impedance particles with prescribed boundary impedance is formulated and its importance in physics and technology is emphasized. It is shown that if this problem is solved then one can easily prepare materials with a desired refraction coefficient, in particular, meta-materials. One can also prepare materials with a desired radiation pattern.

**Keywords:** Scattering theory; Materials science

## Introduction

This is a brief review of the author's work cited in the references. The most important references are [1-4]. The author's theory consists of an elective solution to the many-body scattering problem for small particles of an arbitrary shape, see [5,6]. The practical importance of a method for producing small particles with prescribed boundary impedance is emphasized in [7,8]. Some portions of the text of this paper is borrowed. The authors theory of inverse scattering is presented [9,10]. Monographs present the theory of wave scattering by obstacles and potentials. In section 1 of this paper the basic problem of practical preparing (producing, manufacturing) of small impedance particles is discussed. Wave scattering by many small impedance particles is developed in, where the basic physical assumption is  $a \ll d \ll \lambda$ . Here  $a$  is the characteristic size of the small particles,  $d$  is the minimal distance between neighboring particles, and  $\lambda$  is the wave length in the medium. We refer for this theory [4]. It is proved there that if one prepares many small particles with prescribed boundary impedance and embed these particles (with a specified distribution density, see formula (7) below) into a given material, then one obtains a material whose refraction coefficient approximates any desired refraction coefficient with an arbitrary small error. In particular, one can create meta-materials. How to embed small particles into a given material physicists know.

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Therefore the basic practical problem of preparing small impedance particles with a prescribed boundary impedance is of great interest both technologically and physically. In section 2 of this paper the basic definitions are given. It is explained what a small particle is, what an impedance particle is and what boundary impedance is. In section 3 of this paper a recipe for creating materials with a desired refraction coefficient is formulated [1].

## Basic Definitions

Let  $D_1 \subset R^3$  be a bounded domain with a connected smooth boundary  $S_1$ ,  $D_1' := R^3 \setminus D_1$  be the unbounded exterior domain and  $S^2$  be the unit sphere in  $R^3$ .

Consider the scattering problem:

$$(\Delta^2 + k^2 n_0^2)u = 0 \text{ in } D_1', u_n = \zeta_1 u \text{ on } S_1, u = e^{ika \cdot x} + v, \quad (1)$$

Where  $k > 0$  is the wave number, a constant,  $\alpha \in S^2$  is a unit vector in the direction of the propagation of the incident plane wave  $e^{ika \cdot x}$ ,  $N$  is the unit normal to  $S_1$  pointing out of  $D_1$ ,  $u_N$  is the normal derivative of  $u$ ,  $\zeta_1$  is the boundary impedance,  $Im \zeta_1 \leq 0$ ,  $n_0 > 0$  is the refraction coefficient of the small impedance particle  $D_1$ ,  $n_0$  is a constant, the scattered field  $v$  satisfies the radiation condition

$$v_r - ikv = o\left(\frac{1}{r}\right), r := |x| \rightarrow \infty \quad (2)$$

The scattering amplitude  $A(\beta, \alpha, k) \frac{e^{ikr}}{r} + o\left(\frac{1}{r}\right), r := |x| \rightarrow \infty, \frac{x}{r} = \beta, \quad (3)$

Where  $\alpha, \beta \in S^2, \beta$  is the direction of the scattered wave,  $\alpha$  is the direction of the incident wave. A particle of a characteristic size  $a$  is called small if it is much smaller than the wave length, that is,  $kn_0a \ll 1$ .

The function  $A(\beta, \alpha, k)$  that the solution to the scattering problem does exist and is unique [11-13]. If there are many  $M \gg 1$  small impedance particles  $D_j$ , embedded in a bounded domain  $D$  filled with material whose refraction coefficient is  $n_0$ , then the wave scattering problem can be formulated as follows:

$$(\nabla^2 + k^2 n_0^2)u = 0 \quad \text{In}$$

$$D \setminus \bigcup_{m=1}^M D_m \quad (4)$$

$$u_N = \zeta_m u \text{ on } S_m, 1 \leq m \leq M, \quad (5)$$

$$u = u_0(x) + v, \quad (6)$$

Where the scattered field  $v$  satisfies the radiation condition,  $Im \zeta_m \leq 0, < M$  is the total number of the embedded particles. The incident field  $u_0$  is assumed known. It satisfies equation (4) in  $R^3$  and, as was noted earlier,  $n_0 = 1$  in  $D'$ . For simplicity we assume that all small particles are of the same characteristic size  $a$ .

Let  $G$  be the Green's function of the scattering problem in the absence of the embedded particles. Outside  $D$  the refraction coefficient  $n_0$  is assumed to be equal to 1. Assume that the distribution of small particles is given by the formula

$$N(\Delta) = \frac{1}{a^{2-k}} \int \Delta N(x) dx [1 + o(1)], a \rightarrow 0 \quad (7)$$

Where  $N(\Delta)$  is the number of particles in an arbitrary open subset  $\Delta$  of  $D, N(x) \geq 0$  is a given continuous function,  $k \in [0,1)$  is a number, and the boundary impedance is defined as follows [14]:

$$\zeta_m = \frac{h_m}{a^k} \quad (8)$$

Where  $Im h_m \leq 0, h_m := h(x_m), x_m \in D_m$  is an arbitrary point and  $h(x)$  is a given continuous function in  $D$  this function, number  $k$  and the function  $N(x)$  can be chosen by the experimenter. The field  $u$  in satisfies, as  $a \rightarrow 0$  the following integral equation:

$$u(x) = u_0(x) - c_0 \int D G(x, y) h(y) N(y) dy, c_0 := S/a^2 \quad (9)$$

Where  $S$  is the surface area of a small particle,  $h(x)$  and  $N(x)$  are defined in (7)-(8). For simplicity we assume here that the surface area  $S$  is the same for all small particles. It follows from (9) that the new refraction coefficient in  $D$  which one gets after embedding many small impedance particles [15-17].

$$n(x) = [n_0^2 - c_0 k^{-2} h(x) N(x)]^{1/2} \quad (10)$$

Since  $h(x)$  and  $N(x)$  are at our disposal, one can get by formula (10) any desirable refraction coefficient such that  $Im n^2(x) \geq 0$

Why should the equation (5) make sense physically regardless of the size of the particle?

Because a problem whose solution exists and is unique must have sense physically.

Why should the small impedance particles with a prescribed boundary impedance exist?

Because the particles with  $\zeta = \infty$ , acoustically soft particles, do exist, and the particles with  $\zeta = 0$  acoustically hard particles, do exist, we conclude that small particles with any "intermediate" value of the boundary impedance should also exist.

The problem we raise is:

How can one produce practically (fabricate) such particles?

### Recipe for Creating Material with a Desired Refraction Coefficient

Let us formulate the result in the following theorem:

**Theorem 1:** Given  $n_0^2(x)$  and a bounded domain  $D$  one can create in  $D$  a material with a desired refraction Coefficient  $n(x), Im n^2(x) \geq 0$ , by embedding  $M = O\left(\frac{1}{a^{2-k}}\right)$  small impedance particles according to the distribution law (7). The refraction coefficient  $n_a(x)$  corresponding to a finite  $a$ , approximates the desired refraction coefficient  $n(x)$  in the sense

$$\lim_{a \rightarrow 0} n_a(x) = n(x) \quad (11)$$

The functions  $h(x)$  and  $N(x)$  defined in (7)-(8) are found by the Steps 1, 2 of the Recipe formulated below. Finally, let us discuss briefly the possibility to create material with negative refraction coefficient

one gets  $n(x) < 0$  if the argument of  $n_0^2(x) - k^{-2} C_0 N(x) h(x)$  is equal to  $2\pi$  Assume that  $n_0^2(x) > 0$ . we know that  $k^{-2} C_0 N(x) \geq 0$ .

Let us take  $h(x) = |h(x)|e^{i\varphi}$ , where  $\varphi > 0$  is very small, that is,  $Im h(x) \geq 0$  and  $Im h(x)$  is very small. Equation (9) is uniquely solvable for  $Im h(x) \geq 0$  sufficiently small. For such  $h(x)$  one concludes that the argument  $\arg(n_0^2(x) - k^{-2} C_0 N(x) h(x))$  is very close to  $2\pi$  and the square root in (10) is negative, provided that  $Re(n_0^2(x) - k^{-2} C_0 N(x) h(x)) > 0$  This argument shows that it is possible to create materials with negative refraction coefficient  $n(x)$  by embedding in a given material many small particles with properly chosen boundary impedances.

A recipe for creating materials with a desired refraction coefficient

**Problem 1:** Given a material with known  $n_0(x)$  in a bounded domain  $D$  and  $n_0(x) = 1$  in  $D'$  one wants to create in  $D$  a material with a desired  $n(x)$

**Step 1:** Given  $n_0^2(x)$  and  $n^2(x)$  calculate  $p(x) := k^2 [n_0^2(x) - n^2(x)]$ . This is a trivial step.

**Step 2:** Given  $p(x)$  calculate  $h(x)$  and  $N(x)$  from the equation  $p(x) = C_0 h(x) N(x)$ .

The constant  $C_0$  one can take to be  $C = 4\pi$  if the small particles are balls of radius  $a$ . This we can assume without loss of generality if we are interested in creating materials with a desired refraction coefficient.

There are infinitely many solutions  $h$  and  $N$  to the above equation. For example, one can fix arbitrarily  $N(x) > 0$  in  $D$ ,  $N(x) = 0$  in  $D'$  and find  $h(x) = h_1(x) + ih_2(x)$  by the formulas  $h_1(x) = \frac{p_1(x)}{4\pi N(x)}$ ,  $h_2(x) = \frac{p_2(x)}{4\pi N(x)}$ . Here  $p_1(x) = \text{Rep}(x)$ ,  $p_2(x) = \text{Imp}(x)$ . Note that  $\text{Im}n^2(x) \geq 0$  implies  $\text{Im}h(x) \leq 0$ , so our assumption  $\text{Im}h(x) \leq 0$  is satisfied. Step 2 is also trivial.

**Step 3:** Given  $N(x)$  and  $h(x) = h_1(x) + ih_2(x)$  distribute  $M = O\left(\frac{1}{a^{2-k}}\right)$  small impedance balls of radius  $a$  in the bounded region  $D$  according to the distribution law (7), where  $k \in [0,1)$  is the number (parameter) that can be chosen by the experimenter. Note that condition  $d \gg a$  is satisfied automatically for the distribution law (7). Indeed, if  $d$  is the minimal distance between neighboring particles, then there are at most  $\frac{1}{d^3}$  particles in a cube with the unit side, and since  $D$  is a bounded domain there are at most  $O\left(\frac{1}{d^3}\right)$  of small particles in  $D$ . On the other hand, by the distribution law (7) one has  $N(D) = O\left(\frac{1}{a^{2-k}}\right)$ . Thus  $O\left(\frac{1}{d^3}\right) = O\left(\frac{1}{a^{2-k}}\right)$ . Therefore  $d = O\left(\frac{1}{a^{2-k}}\right)$ . Consequently, condition  $a \ll d \ll \lambda$  is satisfied,  $\lambda = \frac{2\pi}{kn_0}$ , moreover  $\log_{a \rightarrow 0} \frac{a}{d(a)} = 0$ .

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